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Stability of Nontrivial Solutions of the Navier–Stokes System on the Three Dimensional Torus[1]

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This paper examines the stability of nontrivial regular solutions to the Navier–

the estimate for the Stokes system. In particular we prove the stability of unforced two dimensional flows.

Key Words: Navier–Stokes equations; Stokes system; global solution; large data; stability.

1. INTRODUCTION

In the paper we examine the motion of a viscous incompressible fluid described by the Navier–Stokes system on $\mathbf{T}^3 = \mathbf{R}^3/2\pi\mathbf{Z}^3$

$$\begin{aligned} v_t + (v\nabla)\, v - \nu\, \Delta v + \nabla \tilde{q} &= h + f, \\ div\, v &= 0, \\ v|_{t=0} &= w_0 + u_0, \end{aligned} \tag{1.1}$$

where $v = (v_1, v_2, v_3)$ is the velocity, $\tilde{q}$ the pressure, $h + f$ the external force and ν the constant positive viscosity coefficient. System (1.1) is treated as a perturbation of the following problem

$$\begin{aligned} w_t + (w\nabla)\, w - \nu\, \Delta w + \nabla q &= h, \\ div\, w &= 0, \\ w|_{t=0} &= w_0. \end{aligned} \tag{1.2}$$

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Let $u=v-w$ and $p=\tilde{q}-q$. From systems (1.1) and (1.2) we get

$$\begin{aligned} u_t-\nu\,\Delta u+\nabla p&=f-w\,\nabla u-u\,\nabla u-u\,\nabla w,\\ div\; u&=0, \\ u|_{t=0}&=u_0. \end{aligned} \tag{1.3}$$

We require that the solution to (1.2) is regular and global in time such that $w\in W^1_\infty(\mathbf{T}^3\times[0,\infty))$. Moreover, $w=w_1+w_2$ and there exists constant $\mu\geqslant 0$ such that

$$\min\{\Phi(t)-2\mu,0\}\in L_1(0,\infty), \tag{1.4}$$

where

$$\Phi(t)=I_0(2\nu-3\alpha)-6\,\|\nabla w_1(\cdot,t)\|_{L_\infty(\mathbf{T}^3)}-3\alpha^{-1}\,\|w_2(\cdot,t)\|^2_{L_\infty(\mathbf{T}^3)},$$

where α is a positive constant such that $2\nu-3\alpha>0$ and I_0 is the constant from the Poincaré inequality (see (2.3)). These restrictions do not assume the smallness of solution w, and condition (1.4) arises naturally in the energy estimate in Lemma 5.2.

The result of the paper is the following theorem.

THEOREM. *Let* $r\geqslant 2$, $f=\nabla\varphi\in L_{r(loc)}(\mathbf{T}^3\times[0,\infty))$, $u_0\in W^{2-2/r}_r(\mathbf{T}^3)\cap L_2(\mathbf{T}^3)$, $\int_{\mathbf{T}^3}u_0\,dx=0$, $div\,u_0=0$ *and*

$$\sup_{k\in\mathbf{N}}\|f\|_{L_r(\mathbf{T}^3\times[k,k+1])}+\|u_0\|_{W^{2-2/r}_r(\mathbf{T}^3)}+\|u_0\|_{L_2(\mathbf{T}^3)}\leqslant M_0,$$

$$\|u_0\|_{L_2(\mathbf{T}^3)}\leqslant\delta.$$

If $\delta\leqslant\delta_0(M_0)$, *where* $\delta_0(M_0)$ *tends to zero with* M_0 *tending to infinity, then the perturbed solution* $(v,\tilde{q})$ *exists globally in time and the following estimate holds*

$$\|v-w\|_{W^{2,1}_r(\mathbf{T}^3\times[k,k+1])}+\|\nabla\tilde{q}-\nabla q\|_{L_r(\mathbf{T}^3\times[k,k+1])}\leqslant K(M_0), \tag{1.5}$$

for $k\in\mathbf{N}$, *where* $K(M_0)$ *is a function independent of* k *and tends to zero with* M_0 *tending to zero. Moreover, we have*

$$\begin{aligned}&\|v-w\|_{W^{2,1}_r(\mathbf{T}^3\times[k,k+1])}+\|\nabla\tilde{q}-\nabla q\|_{L_r(\mathbf{T}^3\times[k,k+1])}\\ &\quad\leqslant c(e^{-\mu k}\,\|u_0\|_{L_2(\mathbf{T}^3)}+\|f\|_{L_r(\mathbf{T}^3\times[k-1,k+1])})\end{aligned} \tag{1.6}$$

for $k\in\mathbf{N}\backslash\{0\}$.

Remark. It is easily seen that we can reduce problem (3) to the case with $f \equiv 0$. Since $f = \nabla \varphi$ it is possible to transform $p \to p - \varphi$. But in our considerations we will not use such a transformation.

The class of regular solutions which are defined by (1.4) contains regular solutions in two space dimensional case and stationary solutions in a three space dimensional case with periodic boundary conditions and small external forces (see [4, 10]). In the general case only the existence for small data is known [3]. For large data only the existence of weak solutions is known [2]. The stability of special solutions has also been considered in [7, 8].

2. NOTATION

In our considerations we will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$ where $m, n \in R_+ \cup \{0\}$, $r \geqslant 1$ and $Q_T = \mathbf{T}^3 \times (0, T)$ with the norm:

$$
\begin{aligned}
\|u\|^r_{W_r^{m,n}(Q_T)} = & \int_0^T \int_{\mathbf{T}^3} |u(x,t)|^r \, dx \, dt \\
& + \sum_{0 \leqslant |m'| \leqslant [|m|]} \int_0^T \int_{\mathbf{T}^3} |D_x^{m'} u(x,t)|^r \, dx \, dt \\
& + \sum_{|m'| = [|m|]} \int_0^T dt \int_{\mathbf{T}^3} \int_{\mathbf{T}^3} \frac{|D_x^{m'} u(x,t) - D_x^{m'} u(x',t)|^r}{|x - x'|^{3 + r(|m| - [|m|])}} \, dx \, dx' \\
& + \sum_{0 \leqslant |n'| \leqslant [|n|]} \int_0^T \int_{\mathbf{T}^3} |D_t^{n'} u(x,t)|^r \, dx \, dt \\
& + \int_{\mathbf{T}^3} dx \int_0^T \int_0^T \frac{|D_t^{[n]} u(x,t) - D_t^{[n]}(x,t')|^r}{|t - t'|^{1 + r(n - [n])}} \, dt \, dt',
\end{aligned}
\tag{2.1}
$$

where $[\alpha]$ is the integral part of α, $D_x^l = \partial_{x_1}^{l_1} \cdots \partial_{x_s}^{l_s}$ where $l = (l_1, \cdots, l_s)$ is a multiindex.

For these spaces we have the following relations (see [1]). Let $u \in W_r^{m,n}(Q_T)$ then if $\sum_{i=1}^3 (\alpha_i + \frac{1}{r} - \frac{1}{q}) \frac{1}{m} + (\beta + \frac{1}{r} - \frac{1}{q}) \frac{1}{n} < 1$ the following estimate holds

$$
\|D_t^\beta D_x^\alpha u\|_{L_q(Q_T)} \leqslant \varepsilon \, \|u\|_{W_r^{m,n}(Q_T)} + c(\varepsilon) \, \|u\|_{L_2(Q_T)}, \tag{2.2}
$$

where $q \geqslant r \geqslant 2$ and $\varepsilon \in (0, 1)$ and $c(\varepsilon) \to \infty$ with $\varepsilon \to 0$.

In the next section we will apply a corollary from the Marcinkiewicz theorem; thus we have to introduce a base in $L_r(\mathbf{T}^3)$. We define $E: \mathbf{Z} \times [0, 2\pi] \to \mathbf{R}$ such that

$$E(k, z) = \begin{cases} \sin kz & k > 0 \\ \cos kz & k \leqslant 0. \end{cases}$$

It is easily to see that

$$\partial_z E(k, z) = kE(-k, z).$$

Then we can define the base

$$w_{k, l, m}(x_1, x_2, x_3) = E(k, x_1)\, E(l, x_2)\, E(m, x_3)$$

and we have

$$\overline{\operatorname{span}\{\{w_{k, l, m}\}_{k, l, m \in Z^3}\}}^{\|\cdot\|_{L_r}} = L_r(\mathbf{T}^3).$$

Proposition 2.1 (see [6]). *Let* $r > 1$, $f \in L_r(\mathbf{T}^3)$,

$$f(x) = \sum_{(k, l, m) \in \mathbf{Z}^3} f_{k, l, m} w_{k, l, m}(x)$$

and

$$Pf = \sum_{(k, l, m) \in \mathbf{Z}^3} \Phi(k, l, m)\, f_{k, l, m} w_{k, l, m},$$

where $\Phi(n_1, n_2, n_3) = n_i n_j / (n_1^2 + n_2^2 + n_3^2)$ *for* $i, j = 1, 2, 3$. *Then* $P: L_r(\mathbf{T}^3) \to L_r(\mathbf{T}^3)$ *is a bounded operator and*

$$\|Pf\|_{L_r(\mathbf{T}^3)} \leqslant C \|f\|_{L_r(\mathbf{T}^3)}.$$

We also need the Poincare inequality. Let $\int_{\mathbf{T}^3} u(x)\, dx = 0$, then

$$\|u\|_{L_2(\mathbf{T}^3)} \leqslant I_0^{-1} \|\nabla u\|_{L_2(\mathbf{T}^3)}. \tag{2.3}$$

In our considerations we use well-known results as the imbedding or the trace theorems for Sobolev spaces. All constants are denoted by c. By $A, B, C, \ldots$ we denote constants which are fixed in each proof.

3. STOKES SYSTEM

The main tool used in the proof is the estimate for solutions to the Stokes system on torus $\mathbf{T}^3 = \mathbf{R}^3/2\pi\mathbf{Z}^3$

$$\begin{aligned} u_t - \nu\, \Delta u + \nabla p &= f, \\ div\; u &= 0, \\ u|_{t=0} &= u_0. \end{aligned} \tag{3.1}$$

First we examine a system with vanishing initial data

$$\begin{aligned} u_t - \nu\, \Delta u + \nabla p &= f, \\ div\; u &= 0, \\ u|_{t=0} &= 0. \end{aligned} \tag{3.2}$$

LEMMA 3.1. *Let $f \in L_r(\mathbf{T}^3 \times [0, T])$. Then there exists a unique solution of* (3.2) *such that $u \in W_r^{2,1}(\mathbf{T}^3 \times [0, T])$ and $p \in W_r^{1,0}(\mathbf{T}^3 \times [0, T])$ and the following estimate holds*:

$$\|u\|_{W_r^{2,1}(\mathbf{T}^3 \times [0, T])} + \|\nabla p\|_{L_r(\mathbf{T}^3 \times [0, T])} \leqslant C\|f\|_{L_r(\mathbf{T}^3 \times [0, T])}. \tag{3.3}$$

Proof. We look for the solutions in the form

$$u(x, t) = \sum_{(k, l, m) \in Z^3} c_{k, l, m}(t)\, w_{k, l, m}(x),$$

$$p(x, t) = \sum_{(k, l, m) \in Z^3} d_{k, l, m}(t)\, w_{k, l, m}(x)$$

and given function f we can examine in the form

$$f(x, t) = \sum_{(k, l, m) \in Z^3} f_{k, l, m}(t)\, w_{k, l, m}(x).$$

Then system (3.2) takes the form

$$\sum_{(k, l, m) \in Z^3} \dot{c}^1_{k, l, m} w_{k, l, m} + \sum_{(k, l, m) \in Z^3} \nu\, |klm|^2\, c^1_{k, l, m} w_{k, l, m} + \sum_{(k, l, m) \in Z^3} k d_{k, l, m} w_{-k, l, m} = f^1_{k, l, m} w_{k, l, m};$$

$$\sum_{(k, l, m) \in Z^3} \dot{c}^2_{k, l, m} w_{k, l, m} + \sum_{(k, l, m) \in Z^3} \nu\, |klm|^2\, c^2_{k, l, m} w_{k, l, m} + \sum_{(k, l, m) \in Z^3} l d_{k, l, m} w_{k, -l, m} = f^2_{k, l, m} w_{k, l, m};$$

$$\sum_{(k,l,m)\in Z^3} \dot{c}^3_{k,l,m} w_{k,l,m} + \sum_{(k,l,m)\in Z^3} \nu\,|klm|^2\, c^3_{k,l,m} w_{k,l,m}$$
$$+ \sum_{(k,l,m)\in Z^3} m d_{k,l,m} w_{k,l,-m} = f^3_{k,l,m} w_{k,l,m};$$
$$\sum_{(k,l,m)\in Z^3} k c^1_{k,l,m} w_{-k,l,m} + \sum_{(k,l,m)\in Z^3} l c^2_{k,l,m} w_{k,-l,m}$$
$$+ \sum_{(k,l,m)\in Z^3} m c^3_{k,l,m} w_{k,l,-m} = 0, \tag{3.4}$$

where dots denote the differentiation with respect to t and $|klm|^2 = k^2 + l^2 + m^2$.

From (3.4) and the independence of $w_{k,l,m}$ we obtain the following infinity system

$$\begin{aligned}
\dot{c}^1_{k,l,m} + \nu\,|klm|^2\, c^1_{k,l,m} - kd_{-k,l,m} - f^1_{k,l,m} &= 0,\\
\dot{c}^2_{k,l,m} + \nu\,|klm|^2\, c^2_{k,l,m} - ld_{k,-l,m} - f^2_{k,l,m} &= 0,\\
\dot{c}^3_{k,l,m} + \nu\,|klm|^3\, c^1_{k,l,m} - md_{k,l,-m} - f^3_{k,l,m} &= 0,\\
kc^1_{-k,l,m} + lc^2_{k,-l,m} k + mc^3_{k,l,-m} &= 0.
\end{aligned} \tag{3.5}$$

Solving $(3.5)_{1,2,3}$ we get

$$\begin{aligned}
c^1_{k,l,m} &= e^{-\nu\,|klm|^2\, t} \int_0^t e^{\nu\,|klm|^2\, s} (kd_{-k,l,m} + f^1_{k,l,m})\, ds,\\
c^2_{k,l,m} &= e^{-\nu\,|klm|^2\, t} \int_0^t e^{\nu\,|klm|^2\, s} (ld_{k,-l,m} + f^2_{k,l,m})\, ds,\\
c^3_{k,l,m} &= e^{-\nu\,|klm|^2\, t} \int_0^t e^{\nu\,|klm|^2\, s} (md_{k,l,-m} + f^3_{k,l,m})\, ds.
\end{aligned} \tag{3.6}$$

Putting (3.6) into $(3.5)_4$ we get

$$e^{-\nu\,|klm|^2\, t} \int_0^t e^{\nu\,|klm|^2\, s}$$
$$\cdot (k^2 d_{k,l,m} + kf^1_{-k,l,m} + l^2 d_{k,l,m} + lf^2_{k,-l,m} + m^2 d_{k,l,m} + mf_{k,l,-m})\, ds = 0,$$

which gives

$$d_{k,l,m} = -\frac{1}{k^2 + l^2 + m^2} (kf^1_{-k,l,m} + lf^2_{k,-l,m} + mf^3_{k,l,-m}). \tag{3.7}$$

Thus we obtain the pressure

$$p(x, t) = \sum_{(k, l, m) \in Z^3} -\frac{kf^1_{-k, l, m}(t) + lf^2_{k, -l, m}(t) + mf^3_{k, l, -m}(t)}{k^2 + l^2 + m^2} w_{k, l, m}(x). \quad (3.8)$$

Now we can examine $\partial_{x_1} p$ which by (3.8) reads

$$\begin{aligned} \partial_{x_1} p(x, t) = & \sum_{(k, l, m) \in Z^3} \frac{k^2}{k^2 + l^2 + m^2} f^1_{k, l, m}(t)\, w_{k, l, m}(x) \\ & + \sum_{(k, l, m) \in Z^3} \frac{kl}{k^2 + l^2 + m^2} f^2_{-k, -l, m}(t)\, w_{k, l, m}(x) \\ & + \sum_{(k, l, m) \in Z^3} \frac{km}{k^2 + l^2 + m^2} f^3_{-k, l, -m}(t)\, w_{k, l, m}(x). \end{aligned} \quad (3.9)$$

To obtain an estimate we note that multipliers in (3.9)

$$\frac{k^2}{k^2 + l^2 + m^2}, \qquad \frac{kl}{k^2 + l^2 + m^2}, \qquad \frac{km}{k^2 + l^2 + m^2}$$

satisfy the assumptions of Proposition 2.1, hence we get

$$\|p_{x_1}\|_{L_r(\mathbf{T}^3)} \leqslant c\, \|f\|_{L_r(\mathbf{T}^3)}. \quad (3.10)$$

Here we have to note also that in (3.10) we use the following relations for functions from (3.9)

$$\begin{aligned} & \left\| \sum_{(k, l, m) \in Z^3} f^2_{-k, -l, m}(t)\, w_{k, l, m}(x) \right\|_{L_r(\mathbf{T}^3)} \\ & \quad = \left\| \sum_{(k, l, m) \in Z^3} f^2_{k, l, m}(t)\, w_{k, l, m}(x) \right\|_{L_r(\mathbf{T}^3)}. \end{aligned} \quad (3.11)$$

To show (3.11) it is enough to note that $\sin(\alpha + \frac{\pi}{2}) = \cos\alpha$, $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$ and $\sin(-t) = -\sin t$, $\cos -t = \cos t$, then it it is enough to transform $x \to x + \frac{\pi}{2}$ and $x \to -x$. We have the same for $f^3_{-k, l, -m}(t)\, w_{k, l, m}(x)$. So we conclude (3.10) for p_{x_2} and p_{x_3}.

Thus from (3.10) taking the L_r-norm with respect to t we obtain

$$\|\nabla p\|_{L_r(\mathbf{T}^3 \times [0, T])} \leqslant c\, \|f\|_{L_r(\mathbf{T}^3 \times [0, T])}. \quad (3.12)$$

Next we consider the following problem

$$
\begin{aligned}
u_t - \nu\, \Delta u &= f - \nabla p, \\
u|_{t=0} &= 0.
\end{aligned}
\tag{3.13}
$$

By well-known arguments, for instance by potential method, we obtain a unique soliton of (13) such that $u \in W_r^{2,1}(\mathbf{T}^3 \times [0, T])$ and

$$
\|u\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} \leqslant c(\|f\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|\nabla p\|_{L_r(\mathbf{T}^3 \times [0,T])}). \tag{3.14}
$$

Since $div(f - \nabla p) = 0$, we have that the solution of (3.13) satisfies also the following problem

$$
\begin{aligned}
u_t - \nu\, \Delta u &= f - \nabla p, \\
div\, u &= 0, \\
u|_{t=0} &= 0.
\end{aligned}
\tag{3.15}
$$

Thus we obtain a solution of (3.2) and estimate (3.3) comes from (3.12) and (3.14).

LEMMA 3.2. *Let* $r \geqslant 2$, $f \in L_r(\mathbf{T}^3 \times [0, T])$ *and* $u_0 \in W_r^{2-2/r}(\mathbf{T}^3)$ *then there exists a unique solution of* (3.1) *such that* $u \in W_r^{2,1}(\mathbf{T}^3 \times [0, T])$ *and* $p \in W_r^{1,0}(\mathbf{T}^3 \times [0, T])$ *and*

$$
\|u\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \|\nabla p\|_{L_r(\mathbf{T}^3 \times [0,T])} \leqslant c(\|f\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}). \tag{3.16}
$$

Proof. First we introduce an extension of the initial datum u_0 by the following formula:

$$
\bar{u}_0(x, t) = \frac{1}{(4\pi t)^{3/2}} \int_{\mathbf{R}^3} e^{-|x-y|^2/4t} u_0(y)\, dy.
$$

By results from [5, Chap. 4], $\bar{u}_0 \in W_r^{2,1}(\mathbf{T}^3 \times [0, T])$, $\bar{u}_0|_{t=0} = u_0$ and

$$
\|\bar{u}_0\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} \leqslant c\, \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}.
$$

Writing $u = v + \bar{u}_0$, system (3.1) transforms to the following

$$
\begin{aligned}
v_t - \nu\, \Delta v + \nabla p &= f - \bar{u}_{0t} + \nu\, \Delta \bar{u}_0 \equiv f', \\
div\, v &= -div\, \bar{u}_0 \equiv g', \\
v|_{t=0} &= 0,
\end{aligned}
\tag{3.17}
$$

where

$$\begin{aligned} \|f'\|_{L_r(\mathbf{T}^3\times[0,T])} &\leqslant \|f\|_{L_r(\mathbf{T}^3\times[0,T])} + c\,\|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}, \\ \|g'\|_{W_r^{1,0}(\mathbf{T}^3\times[0,T])} &\leqslant c\,\|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}. \end{aligned} \tag{3.18}$$

Solving the following equation

$$\Delta\Phi = g', \tag{3.19}$$

we get

$$\begin{aligned} \|\nabla\Phi\|_{W_r^{2,0}(\mathbf{T}^3\times[0,T])} &\leqslant c\,\|g'\|_{W_r^{1,0}(\mathbf{T}^3\times[0,T])}, \\ \|\nabla\Phi_t\|_{L_r(\mathbf{T}^3\times[0,T])} &\leqslant c\,\|\partial_t\bar{u}_0\|_{L_r(\mathbf{T}^3\times[0,T])}. \end{aligned} \tag{3.20}$$

Taking $v = u' + \nabla\Phi$ we obtain

$$\begin{aligned} u'_t - \nu\,\Delta u + \nabla p &= f' - \nabla\Phi_t + \nu\,\Delta\nabla\Phi \equiv f'', \\ div\,u' &= 0, \\ u'|_{t=0} &= 0. \end{aligned} \tag{3.21}$$

Applying Lemma 3.1 to (3.21) we get a solution with the following estimate

$$\|u'\|_{W_r^{2,1}(\mathbf{T}^3\times[0,T])} + \|\nabla p\|_{L_r(\mathbf{T}^3\times[0,T])} \leqslant c\,\|f''\|_{L_r(\mathbf{T}^3\times[0,T])}. \tag{3.22}$$

Summing up we have obtained the solution to (3.1) given by $u = \bar{u}_0 + \nabla\Phi + u'$ and from (3.18), (3.20), and (3.22) we conclude (3.16).

4. LOCAL EXISTENCE

Next we prove a local existence of the solutions.

LEMMA 4.1. *Let $f \in L_r(\mathbf{T}^3\times[0,T])$, $u_0 \in W_r^{2-2/r}(\mathbf{T}^3)$. Then there exists $T_0 > 0$ such that for all $T \leqslant T_0$ system* (1.3) *has a unique solution such that $u \in W_r^{2,1}(\mathbf{T}^3\times[0,T])$, $p \in W_{r(loc)}^{1,0}(\mathbf{T}^3\times[0,T])$ and the following estimate holds*

$$\begin{aligned} &\|u\|_{W_r^{2,1}(\mathbf{T}^3\times(0,T))} + \|\nabla p\|_{L_r(\mathbf{T}^3\times(0,T))} \\ &\qquad \leqslant C(T_0)(\|f\|_{L_r(\mathbf{T}^3\times(0,T))} + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}). \end{aligned} \tag{4.1}$$

Proof. We construct a sequence of approximations $\{u_m, p_m\}_{m=1}^{\infty}$ defined by the following relation,

$$
\begin{aligned}
u_{m,t} - \nu \Delta u_m + \nabla p_m &= -u_{m-1} \nabla u_{m-1} + f - w \nabla u_{m-1} - u_{m-1} \nabla w, \\
div\, u_m &= 0, \qquad\qquad (4.2)\\
u_m|_{t=0} &= u_0,
\end{aligned}
$$

for $m > 1$ and we put $u_1 = 0$ and $p_1 = 0$.

By Lemma 3.2 we get the estimate for the solution of (4.2)

$$
\begin{aligned}
&\|u_m\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \|\nabla p\|_{L_r(\mathbf{T}^3 \times [0,T])} \\
&\quad \leqslant A(\|f\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)} + \|u_{m-1} \nabla u_{m-1}\|_{L_r(\mathbf{T}^3 \times [0,T])} \\
&\qquad + \|w \nabla u_{m-1}\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|u_{m-1} \nabla w\|_{L_r(\mathbf{T}^3 \times [0,T])}). \qquad (4.3)
\end{aligned}
$$

For $r \geqslant 3$, $W_r^{2-2/r}(\mathbf{T}^3) \subset\subset L_{3r}(\mathbf{T}^3)$ and $W_r^{1-2/r}(\mathbf{T}^3) \subset\subset L_{(3/2)\,r}(\mathbf{T}^3)$ we have the following estimate

$$
\begin{aligned}
&\sup_{t \leqslant T} \|\nabla u_{m-1}(\cdot, t)\|_{L_{(3/2)\,r}(\mathbf{T}^3)} + \sup_{t \leqslant T} \|u_{m-1}(\cdot, t)\|_{L_{3r}(\mathbf{T}^3)} \\
&\quad \leqslant c \sup_{t \leqslant T} \|u_{m-1}(\cdot, t)\|_{W_r^{2-2/r}(\mathbf{T}^3)} \\
&\quad \leqslant \bar{\alpha}(\|u_{m-1}\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}), \qquad (4.4)
\end{aligned}
$$

where $\bar{\alpha}$ does not depend on T. By (4.4) and the Hölder inequality with $\frac{1}{r} = \frac{1}{3r} + \frac{2}{3r}$ we have the relation

$$
\begin{aligned}
&\|u_{m-1} \nabla u_{m-1}\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|w \nabla u_{m-1}\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|u_{m-1} \nabla w\|_{L_r(\mathbf{T}^3 \times [0,T])} \\
&\quad \leqslant BT^{1/r}(\|u_{m-1}\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \|u_{m-1}\|^2_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \\
&\qquad + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)} + \|u_0\|^2_{W_r^{2-2/r}(\mathbf{T}^3)}). \qquad (4.5)
\end{aligned}
$$

If $2 \leqslant r \leqslant 3$ to have (4.5) we use the parabolic imbeddings.

Let us assume that

$$
\begin{aligned}
&\|u_k\|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \|\nabla p_k\|_{L_r(\mathbf{T}^3 \times [0,T])} \\
&\quad \leqslant 4A(\|f\|_{L_r(\mathbf{T}^3 \times [0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}) \equiv M \qquad (4.6)
\end{aligned}
$$

for $k = 1, \cdots, m-1$.

By (4.3) and (4.5) we see that we can choose T so small that (4.6) would be satisfied for $k = m$. Since $u_1 = 0$ and $p_1 = 0$ by the induction we have proved (4.6) for $k \in \mathbf{N}$.

From (4.2) we get the system on $U_m = u_m - u_{m-1}$ and $P_m = p_m - p_{m-1}$ which reads

$$\begin{aligned} U_{m,t} - \nu \Delta U_m + \nabla P_m = & -w \nabla U_{m-1} \\ & - U_{m-1} \nabla w - u_{m-1} \nabla U_{m-1} - U_{m-1} \nabla u_{m-1}, \\ div\, U_{m-1} = & \, 0, \\ U_m|_{t=0} = & \, 0. \end{aligned} \tag{4.7}$$

By Lemma 3.2 we obtain the estimate

$$\begin{aligned} & \| U_m \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \| \nabla P_m \|_{L_r(\mathbf{T}^3 \times [0,T])} \\ & \quad \leqslant A(\| u_{m-1} \nabla U_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} + \| U_{m-1} \nabla u_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} \\ & \qquad + \| w \nabla U_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} + \| U_{m-1} \nabla w \|_{L_r(\mathbf{T}^3 \times [0,T])}). \end{aligned} \tag{4.8}$$

By the same argument as in (4.5) we have

$$\begin{aligned} & \| w \nabla U_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} + \| U_{m-1} \nabla w \|_{L_r(\mathbf{T}^3 \times [0,T])} \\ & \quad \leqslant c T^{1/r} \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])}. \end{aligned} \tag{4.9}$$

By (2.2) with $r \geqslant 2$ we have $W_r^{2,1}(\mathbf{T}^3 \times [0,T]) \subset\subset L_{3r}(\mathbf{T}^3 \times [0,T])$ and $D_x W_r^{2,1}(\mathbf{T}^3 \times [0,T]) \subset\subset L_{(3/2)r}(\mathbf{T}^3 \times [0,T])$, hence applying the Hölder inequality ($\frac{1}{r} = \frac{1}{3r} + \frac{2}{3r}$) we get

$$\begin{aligned} & \| U_{m-1} \nabla u_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} + \| u_{m-1} \nabla U_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])} \\ & \quad \leqslant c(\varepsilon \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + c(\varepsilon) \| U_{m-1} \|_{L_r(\mathbf{T}^3 \times [0,T])}). \end{aligned} \tag{4.10}$$

Using (4.4),

$$\leqslant c(\varepsilon \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + c(\varepsilon)\, T^{1/r} \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])}),$$

where ε is defined as in (2.2).

From (4.8), (4.9), and (4.10) we get

$$\begin{aligned} & \| U_m \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + \| \nabla P_m \|_{L_r(\mathbf{T}^3 \times [0,T])} \\ & \quad \leqslant C(\varepsilon \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])} + c(\varepsilon)\, T^{1/r} \| U_{m-1} \|_{W_r^{2,1}(\mathbf{T}^3 \times [0,T])}). \end{aligned} \tag{4.11}$$

Taking ε and T small enough, we conclude that $U_m \to 0$ in $W_r^{2,1}(\mathbf{T}^3 \times [0,T])$ and $\nabla P_m \to 0$ in $L_r(\mathbf{T}^3 \times [0,T])$. Thus $\{u_m, p_m\}_{m=1}^{\infty}$ is convergent to a solution of (1.3) and (4.1) comes from (4.6). The proof of Lemma 4.1 is ended.

5. GLOBAL EXISTENCE

LEMMA 5.1. *We have*

$$\int_{\mathbf{T}^3} u(x,t)\,dx = 0. \tag{5.1}$$

Proof. Integrating $(1.3)_1$ over $\mathbf{T}^3$, and using $(1.2)_2$, $(1.3)_2$, and assumptions $\int_{\mathbf{T}^3} u_0\,dx = 0$ and $f = \nabla\varphi$, we get (5.1).

LEMMA 5.2. *For the local solution on interval* $[0, T]$ *we have the following estimate,*

$$\|u(\cdot,t)\|_{L_2(\mathbf{T}^3)} \leqslant \bar{A} \exp\{-\mu t\}\ \|u_0\|_{L_2(\mathbf{T}^3)},$$

where $\bar{A}$ *does not depend on* T *and* μ *is defnied by* (1.4).

Proof. Multiplying $(1.3)_1$ by u and integrating over $\mathbf{T}^3$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int u^2\,dx + \nu\int |\nabla u|^2\,dx = &-\int \nabla p\cdot u + \int f\cdot u\,dx\\ &-\int (u\nabla)\,w\cdot u\,dx - \int (u\nabla)\,u\cdot u\,dx\\ &-\int (w\,\nabla)\,u\cdot u\,dx.\end{aligned} \tag{5.2}$$

First, fourth, and fifth terms vanish by $(1.2)_2$ and $(1.3)_2$. The second term vanishes by the assumption that f is potential ($f = \nabla\varphi$). Thus we obtain

$$\frac{d}{dt}\int u^2\,dx + 2\nu\int |\nabla u|^2\,dx \leqslant 6\,\|\nabla w_1\|_{L_\infty(\mathbf{T}^3)}\int u^2\,dx + 2\int u\,\nabla w_2\cdot u\,dx,$$

but

$$\begin{aligned}2\left|\int u\,\nabla w_2\cdot u\,dx\right| &= 2\left|\int u\,\nabla u\cdot w_2\,dx\right|\\ &\leqslant 3\alpha\int |\nabla u|^2\,dx + 3\alpha^{-1}\,\|w_2(\cdot,t)\|^2_{L_\infty(\mathbf{T}^3)}\int u^2\,dx,\end{aligned}$$

where $2\nu - 2\alpha > 0$.

Hence, in the view of (2.3) and Lemma 5.1, we get

$$\frac{d}{dt}\int u^2\,dx+\Phi(t)\int u^2\,dx\leqslant 0, \tag{5.3}$$

where

$$\Phi(t)=I_0(2\nu-3\alpha)-6\,\|\nabla w_1(\cdot,t)\|_{L_\infty(\mathbf{T}^3)}-3\alpha^{-1}\,\|w_2(\cdot,t)\|^2_{L_\infty(\mathbf{T}^3)}.$$

By the Gronwall inequality and (1.4), from (5.3) we obtain

$$\|u(\cdot,t)\|^2_{L_2(\mathbf{T}^3)}\leqslant \exp\left\{-\int_0^t\Phi(s)\,ds\right\}\cdot\|u_0\|^2_{L_2(\mathbf{T}^3)}.$$

By assumption (1.4) we get the thesis of the lemma.

Lemma 5.3. *For a local solution on time interval $[0,T]$ we have the following estimate*

$$\begin{aligned}&\|u\|_{W_r^{2,1}(O_k)}+\|\nabla p\|_{L_r(O_k)}\\&\quad\leqslant C(\|f\|_{L_r(O_{k-1}\cup O_k)}+\|u_0\|_{L_2(\mathbf{T}^3)}+\|u_0\|_{W_r^{2-2/r}(\mathbf{T}^3)}),\end{aligned} \tag{5.4}$$

where $O_k=\mathbf{T}^3\times[kT_1,(k+1)\,T_1]$, $T_1\leqslant T$, and C is independent of k, if $\|u_0\|_{L_2(\mathbf{T}^3)}$ is small enough.

Proof. We introduce a smooth function $\zeta_k\colon\mathbf{R}_+\to\mathbf{R}$ such that

$$\zeta_k(t)=\begin{cases}1 & \text{for}\quad t\geqslant kT_1\\ 0 & \text{for}\quad t\leqslant(k-1)\,T_1\end{cases}$$

for $k\in\mathbf{N}\setminus\{0\}$, $0\leqslant\zeta\leqslant 1$, $\zeta'\geqslant 0$ and $|\zeta'|\leqslant 2/T_1$.

Multiplying $(1.3)_1$ by ζ_k and denoting $U^k=\zeta_k u$, $P^k=\zeta p$ we get

$$\begin{aligned}U^k_t-\nu\,\Delta U^k-\nabla P^k&=\zeta_k f-U^k\,\nabla w-w\,\nabla U^k-u\,\nabla U^k+\zeta'_k u,\\ \operatorname{div}\,U^k&=0,\\ U^k|_{t=(k-1)\,T_1}&=0.\end{aligned} \tag{5.5}$$

By Lemma 3.2 we obtain

$$\begin{aligned}&\|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)}+\|\nabla P^k\|_{L_r(O_{k-1}\cup O_k)}\\&\quad\leqslant A(\|f\|_{L_r(O_{k-1}\cup O_k)}+\|U^k\,\nabla w\|_{L_r(O_{k-1}\cup O_k)}+\|w\,\nabla U^k\|_{L_r(O_{k-1}\cup O_k)}\\&\qquad+\|u\,\nabla U^k\|_{L_r(O_{k-1}\cup O_k)}+\|\zeta'_k u\|_{L_r(O_{k-1}\cup O_k)}).\end{aligned} \tag{5.6}$$

Applying (2.2) we estimate the unknown terms of the r.h.s of (5.6)

$$\begin{aligned}
&\|U^k \nabla w\|_{L_r(O_{k-1}\cup O_k)} \\
&\qquad \leqslant \|\nabla w\|_{L_\infty} (\varepsilon \, \|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon) \, \|U^k\|_{L_2(O_{k-1}\cup O_k)}), \\
&\|w \nabla U^k\|_{L_r(O_{k-1}\cup O_k)} \\
&\qquad \leqslant \|w\|_{L_\infty} (\varepsilon \, \|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon) \, \|U^k\|_{L_2(O_{k-1}\cup O_k)}), \qquad (5.7) \\
&\|u \nabla U^k\|_{L_r(O_{k-1}\cup O_k)} \leqslant \|U^{k-1}\|_{L_{3r}(O_{k-1})} (\varepsilon \, \|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} \\
&\qquad + c(\varepsilon) \, \|U^k\|_{L_2(O_{k-1}\cup O_k)}) + \|U^k \nabla U^k\|_{L_r(O_k)}, \\
&\|\zeta'_k u\|_{L_r(O_{k-1}\cup O_k)} \leqslant \varepsilon \, |\zeta'_k| \, \|U^{k-1}\|_{W_r^{2,1}(O_{k-1})} + |\zeta'_k| \, c(\varepsilon) \, \|u\|_{L_2(O_{k-1})}.
\end{aligned}$$

In $(5.7)_3$ we have applied the same argument as in Lemma 4.1. By (1.4) we have boundedness of $\|\nabla w\|_{L_\infty(\mathbf{T}^3\times[0,T])}$ and $\|w\|_{L_\infty(\mathbf{T}^3\times[0,T])}$; hence we can choose ε so small that we get, applying (5.7) to (5.6), the following estimate

$$\begin{aligned}
&\|U^k\|_{W_r^{2,1}(O_k)} + \|\nabla P^k\|_{L_r(O_k)} \\
&\quad \leqslant A(\|U^k \nabla U^k\|^2_{L_r(O_k)} + \|f\|_{L_r(O_{k-1}\cup O_k)} + \varepsilon \, |\zeta'_k| \, \|U^{k-1}\|_{W_r^{2,1}(O_{k-1})} \\
&\qquad + (c(\varepsilon) \, \|U^{k-1}\|_{W_r^{2,1}(O_{k-1})} \, |\zeta'_k| + c(\varepsilon) \, W) \, \|u\|_{L_2(O_{k-1}\cup O_k)}), \qquad (5.8)
\end{aligned}$$

where $W = \|\nabla w\|_{L_\infty(\mathbf{T}^3\times[0,T])} + \|w\|_{L_\infty(\mathbf{T}^3\times[0,T])}$.

By the same argument as in (4.4) we obtain

$$\begin{aligned}
\|U^k \nabla U^k\|_{L_r(O_k)} &\leqslant \|U^k\|_{L_{3r}} \, \|\nabla U^k\|_{L_{(3/2)r}(O_k)} \\
&\leqslant \sigma \, \|U^k\|^2_{W_r^{2,1}(O_k)} + c(\sigma) \, \|u\|^2_{L_\infty(kT_1,(k+1)T_1; L_2(R^3))}, \qquad (5.9)
\end{aligned}$$

where $\sigma \in (0, 1)$.

Let

$$X_m = \|U^m\|_{W_r^{2,1}(O_m)} + \|\nabla P^m\|_{L_r(O_m)},$$
$$F = \sup_m B \, \|f\|_{L_r(O_{m-1}\cup O_m)}.$$

We prove estimate (5.4) by the induction. For $k=0$, U^0 and P^0 are defined as the local solution described by Lemma 4.1. We assume that $X_{k-1} \leqslant M$, where $M \geqslant 16F$.

Then from (5.8) and (5.9) we obtain

$$\begin{aligned}
X_k \leqslant F + B\varepsilon M + c(\varepsilon)(M+W) \, \|u\|_{L_2(O_{k-1}\cup O_k)} \\
+ c(\sigma) \, \|u\|^2_{L_\infty((k-1)T_1, kT_1; L_2(\mathbf{T}^3))} + B\sigma X_k^2 \equiv \alpha + \beta X_k^2.
\end{aligned}$$

Thus to obtain an uniformly boundedness we have to have

$$1-4\alpha\beta>0 \qquad \text{and} \qquad \frac{1-\sqrt{1-4\alpha\beta}}{2\beta}\leqslant M \tag{5.10}$$

which will give $X_k \leqslant M$. Taking σ, ε and $\|u_0\|_{L_2(\mathbf{T}^3)}$ sufficiently small, by Lemma 5.2, we get the first condition of (5.10). To obtain the second we examine

$$\frac{1-\sqrt{1-4\alpha\beta}}{2\beta}\leqslant 2\alpha.$$

And we take the following relations $F\leqslant\frac{1}{16}M$, ε so small that $B\varepsilon\leqslant\frac{1}{16}$ and $c(\varepsilon)(M+W)\,\|u\|_{L_2(O_{k-1}\cup O_k)}\leqslant\frac{1}{8}M$, $c(\sigma)\,\|u\|^2_{L_\infty((k-1)\,T_1,\,kT_1;\,L_2(\mathbf{T}^3))}\leqslant\frac{1}{4}M$ which by Lemma 5.2 can be satisfied if $\|u_0\|_{L_2(\mathbf{T}^3)}$ is small enough. This way we get $X_k\leqslant M$. By the induction we have proved Lemma 5.3.

By Lemma 5.3 we obtain an uniformly boundedness of solutions independent of time. Thus by Lemma 4.1 we can prolong the solution in time. Hence we have obtained the global solution and inequality (1.5) comes from (5.4). To obtain (1.6) we prove the lemma.

LEMMA 5.4. *For the solution of* (1.3) *we have the following estimate*

$$\begin{aligned}&\|v-w\|_{W_r^{2,1}(\mathbf{T}^3\times[k,k+1])}+\|\nabla\tilde{q}-\nabla q\|_{L_r(\mathbf{T}^3\times[k,k+1])}\\&\qquad\leqslant c(e^{-\mu k}\,\|u_0\|_{L_2(\mathbf{T}^3)}+\|f\|_{L_r(\mathbf{T}^3\times[k-1,k+1])})\end{aligned}$$

for $k\in\mathbf{N}\setminus\{0\}$.

Proof. We define a smooth function $\eta_l\colon\mathbf{R}_+\to\mathbf{R}$ such that

$$\eta_l(t)=\begin{cases}1 & \text{for}\quad t\geqslant t_0+l\\[1ex] 0 & \text{for}\quad t\leqslant t_0+\dfrac{l}{2}\end{cases}$$

and $0\leqslant\eta\leqslant1$, $|D\eta|\leqslant\frac{3}{l}$, $l\in(0,1]$.

Now we repeat considerations from the proof of Lemma 5.3. We multiply (1.3) by η_l, applying Lemma 3.2 we estimate functions $U_l=\eta_l u$ and $P_l=\eta_l p$. This way, using Lemma 5.2, we get

$$\begin{aligned}&\|U_l\|_{W_r^{2,1}(\mathbf{T}^3\times[t_0,t_0+2])}+\|\nabla P_l\|_{L_r(\mathbf{T}^3\times[t_0,t_0+2])}\\&\qquad\leqslant A(\|f\|_{L_r(\mathbf{T}^3\times[t_0,t_0+2])}+Be^{-\mu t_0}\,\|u_0\|_{L_2(\mathbf{T}^3)}+\frac{3}{l}\,\|U_{l/2}\|_{L_r(\mathbf{T}^3\times[t_0,t_0+2])}).\end{aligned}\tag{5.11}$$

To obtain (5.11) we repeat all steps ((5.6), (5.7), (5.8)) of considerations for (5.5) omitting $(5.7)_3$.

From [1, Chap. 18] we have the following interpolation inequality

$$\|u\|_{L_r(\mathbf{T}^3)} \leqslant \varepsilon \|u\|_{W_r^2(\mathbf{T}^3)} + D_1 \varepsilon^{-3/2((1/2)-(1/r))} \|u\|_{L_2(\mathbf{T}^3)}, \tag{5.12}$$

where $\varepsilon \in (0, 1)$. Hence from (5.12) and Lemma 5.2 we get

$$\begin{aligned} &\frac{3}{l} \|U_{l/2}\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\quad \leqslant \varepsilon_0 \|U_{l/2}\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\qquad + D_2 l^{-1} (\varepsilon_0 l)^{-3/2((1/2)-(1/r))} \|u\|_{L_\infty(t_0, t_0+2; L_2(\mathbf{T}^3))}. \end{aligned} \tag{5.13}$$

Applying (5.13) to (5.11) we get

$$\begin{aligned} &\|U_l\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\quad \leqslant \varepsilon_0 \|U_{l/2}\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} + D_3 l^{-1} (\varepsilon_0 l)^{-3/2((1/2)-(1/r))} e^{-\mu t_0} \|u_0\|_{L_2(\mathbf{T}^3)} \\ &\qquad + A(\|f\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} + B e^{-\mu t_0} \|u_0\|_{L_2(\mathbf{T}^3)}). \end{aligned} \tag{5.14}$$

Repeating (5.14) for $l = 1$ we obtain

$$\begin{aligned} &\|U_1\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} + \|\nabla P_1\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\quad \leqslant \varepsilon_0^k \|U_{1/2^k}\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\qquad + e^{-\mu t_0} D_4 \|u_0\|_{L_2(\mathbf{T}^3)} (1 + \varepsilon_0 2^{7/4 - 3/3r} + \cdots + (\varepsilon_0 2^{7/4 - 3/3r})^{k-1}) \\ &\qquad + A(\|f\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} + B e^{-\mu t_0} \|u_0\|_{L_2(\mathbf{T}^3)})(1 + \varepsilon_0 + \cdots + \varepsilon_0^{k-1})). \end{aligned} \tag{5.15}$$

If we take ε_0 so small that $\varepsilon_0 \leqslant 2^{-((7/4)-(3/3r))}$ then we obtain

$$\begin{aligned} &\|U_1\|_{W_r^{2,1}(\mathbf{T}^3 \times [t_0, t_0+2])} + \|\nabla P_1\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} \\ &\quad \leqslant C(\|f\|_{L_r(\mathbf{T}^3 \times [t_0, t_0+2])} + e^{-\mu t_0} \|u_0\|_{L_2(\mathbf{T}^3)}). \end{aligned} \tag{5.16}$$

From (5.16) we conclude (1.6). One can find similar considerations in [8]. The proof of the theorem is ended.

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